DIFFERENTIAL METHODS APPLIED TO THE SOLUTION OF TWO-SIDED ALLOCATION PROBLEMS

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Submitted in partial fulfillment of
the requirements for the degree of

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IN

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ABSTRACT

A procedure is developed which can be used to obtain the solution of a two-sided allocation problem with payoff of the form $I = \int_a^b K(x,y,t)dt$ if K is concave in x(t), i.e. $K_{xx} < 0$, and convex in y(t) for all t in the interval [a,b]. A second degree polynomial example is explained in detail to illustrate the procedure. The computer program and a sample output for this example constitute the Appendix. Next an exponential search problem is discussed. Then in conclusion a general application of the method is outlined.

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1. Introduction.

The purpose of this thesis is to investigate a method for obtaining the optimum solutions of two-sided allocation problems. A two-sided allocation problem is one in which there are two sides allocating resources with opposing interests. Generally one side can optimize its position by minimizing some factor, such as $\int_a^b K\,dt$ where each side has some control of K, which the opposing side would like to maximize, or vice versa.

The problem of maximizing (or minimizing) a functional $\int_a^b K[y(t)]\,dt$ over a certain class of functions $\{y(t)\}$ has been studied by use of various methods. Among these are the use of Lagrange multipliers[2, 9], techniques of calculus of variations[1, 8], and methods of control theory[3] such as the differential methods used in this paper.

There appears to have been little application of these methods to the two-sided allocation problem where K is a function of two functions, x(t) and y(t). Although the theory of games has application to this type of problem if K(x,y,t) is a real valued function that is convex in y, i.e. $K_{yy} \geq 0$, and concave in x[6], it does not provide a method for obtaining a solution. The subscript indicates partial differentiation with respect to the corresponding variable.

Until the last five years there did not exist a method which could be used to obtain the solution of optimization problems subject to constraints on resources. However, differential methods such as those used by Faulkner[3] have been

developed recently which can solve such problems. A method for determining solutions by differential equations using an analog computer has been developed by Kose[4].

A version of the differential methods given by Faulkner [3] will be used in this thesis. These methods not only seem simpler to grasp and apply to the particular functionals considered, but they provide a method for solving either maximization, minimization, or minimization-maximization (saddle value) problems with a speed and simplicity that none of the others (with the possible exception of Kose[4]) seem to have.

2. Basic principles.

Consider a functional of the form

(1) $$I = \int_a^b K\, dt$$

where: $K = K(x,y,t)$ is a real valued function; $x = x(t)$ is a function to be found which maximizes (1) subject to

(2) $$x \geq 0 \quad \text{and} \quad \int_a^b x\, dt = 1;$$

and $y = y(t)$ is a function to be found which minimizes (1) subject to

(3) $$y \geq 0 \quad \text{and} \quad \int_a^b y\, dt = 1.$$

A function x, or y, which is piecewise continuous in [a,b] and satisfies (2), or (3), will be called admissible. If the total resources available to each side are not unity, they can easily be normalized to comply with (2) and (3), making the functions $x(t)$ and $y(t)$ density functions which indicate relative allocation of resources.

It will be convenient to introduce a new functional, J. Let $H = K + \lambda x + \mu y$ where λ and μ are constants (Lagrange multipliers) to be found. Let

(4) $$J = \int_a^b H\, dt.$$

We see that if x and y are admissible functions, then J differs from I by the number $\lambda + \mu$.

Now if y were a given function we could get a maximum value for J,

$$J(x_0,y) = \underset{x}{\text{maximum}}\ J(x,y)$$

by maximizing the Hamiltonian H, subject to the constraint $x \geq 0$, for each value of t on the interval [a,b]; however the function $x(t)$ might not be admissible. Also, for given

$x(t)$ we could get a minimum

$$J(x,y_0) = \underset{y}{\text{minimum}}\ J(x,y).$$

It should be pointed out that the function x_0 above depends on both $y(t)$ and the parameter λ, and y_0 depends upon $x(t)$ and μ. Observe that also

$$(5) \qquad J(x,y_0) \leq J(x_0,y_0) \leq J(x_0,y)$$

since by the way it was obtained x_0 maximizes $J(x,y)$ for any given y, even y_0, and y_0 minimizes $J(x,y)$ for any given x, even x_0.

If the functions x_0 and y_0 are admissible, we will call them the min-max strategies (x^*,y^*) and we will call $J(x^*,y^*)$ the min-max solution. Hence, to solve a min-max problem we want to determine, for some λ and μ initially unknown, $J(x_0,y_0)$ subject to the restrictions (2) and (3) on $x(t)$ and $y(t)$. Any pair of admissible functions (x^*,y^*) which furnishes the min-max solution of (4) will also provide the min-max solution of (1) since

$$(6) \qquad J = \int_a^b H\,dt = \int_a^b (K + \lambda x + \mu y)dt = I + (\lambda + \mu).$$

Let us assume that K has continuous second derivatives. Then notice that an absolute maximum (or minimum) which is not a stationary point can only occur at $x = 0$ (or $y = 0$) for admissible functions. If an absolute maximum occurred for $x = x_1 > 0$, then any larger value of x, say $x = x_1 + \epsilon$, would yield a larger value of H since $H_x > 0$ at x_1. A similar argument applies to minimization over y. Therefore for a min-max solution to exist, the functional $K(x,y,t)$ must not be monotone increasing in x or monotone decreasing in y.

If the second derivatives of K are continuous, then the following conditions exist. To get $\max_x H$ at a point on the interval [a,b], either there is a relative maximum for which

(7) $$H_x = K_x + \lambda = 0$$

or else, if the maximum occurs on the boundary then

(8) $$x = 0.$$

Similarly for $\min_y H$, either there is a relative minimum for which

(9) $$H_y = K_y + \mu = 0$$

or else, if the minimum occurs on the boundary then

(10) $$y = 0.$$

An indication of a boundary situation will be negative values of x or y as solutions of equations (7) and (9).

A min-max problem may also be called a saddle value problem if the integrand is strictly concave in x ($K_{xx} < 0$) and strictly convex in y ($K_{yy} > 0$) on the interval [a,b]. According to Kuhn and Tucker[5] the conditions

(11) $$\begin{array}{lll} H_x(x_0,y_0) \leq 0, & H_x(x_0,y_0)x_0 = 0, & x_0 \geq 0 \\ H_y(x_0,y_0) \geq 0, & H_y(x_0,y_0)y_0 = 0, & y_0 \geq 0 \end{array}$$

are necessary in order that (x_0,y_0) provide a solution for the saddle value problem, which neatly summarizes the previous two paragraphs.

If the function K is not concave in x and convex in y the possibility of multiple solutions of (7) and (9) arises. In this case the difficulty of computation increases consider-

ably. Each possible pair of roots (x,y) must be checked to determine the x which maximizes H and the y which minimizes H. This operation must be repeated for each value of the parameter t on the interval [a,b].

Only the simple case where $K(x,y,t)$ is strictly concave in x, $K_{xx} < 0$, and strictly convex in y, $K_{yy} > 0$, over the domain [a,b] of the parameter t will be considered further. Differentials will be defined and a Newton-Raphson iteration set up to determine the desired solution.

While we had no trouble with convergence, this is often a problem when using a Newton-Raphson iteration procedure.

3. Computational procedure.

In this section we will set up a Newton-Raphson iteration for the solution. To start, let us guess values for the Lagrange multipliers λ and μ and compute a trial solution for x(t) and y(t) using equations (7), (8), (9) and (10). Usually it will not be an admissible solution since

$$\int_a^b x\,dt = X \neq 1 \qquad \text{and} \qquad \int_a^b y\,dt = Y \neq 1.$$

But we shall see that if the solution thus found is, by rare good luck, admissible, then we have the min-max solution. This follows directly from (5) since any trial solution is (x_0, y_0) for the particular values of λ and μ chosen.

Let us now set up a routine for correcting the values of (X,Y) obtaining a sequence approaching (1,1). Suppose we consider the given trial solution and neighboring values of λ and μ, say $\lambda + d\lambda$ and $\mu + d\mu$. Every optimal solution must satisfy $H_x(x_0,y_0)x_0 = 0$ and $H_y(x_0,y_0)y_0 = 0$ from equations (11). Generally only one of the terms in each product is zero at any point on the interval [a,b]. If these zero terms are constrained to remain zero as we change x, y, λ and μ, by small amounts δx, δy, $d\lambda$ and $d\mu$, then x and y must always maximize and minimize, respectively, H since the requirements (11) remain satisfied. We must change x, y, λ and μ in such a way, satisfying the above, that the values X and Y approach unity. In order to calculate

$$dX = \int_a^b \delta x\,dt \qquad \text{and} \qquad dY = \int_a^b \delta y\,dt$$

let us divide the interval [a,b] into four subsets of intervals T_0, T_1, T_2 and T_3. On the intervals in T_0 the min-max

values of x and y are obtained from (7) and (9), i.e. $x > 0$ and $y > 0$; on those in T_1 we use (7) and (10), i.e. $x > 0$ and $y = 0$; on those in T_2 we use (8) and (9), i.e. $x = 0$ and $y > 0$; and on those in T_3 we use (8) and (10), i.e. $x = 0$ and $y = 0$. The resulting variations $\delta x(t)$ and $\delta y(t)$ will be given in the intervals of T_0 by

$$(12) \qquad \begin{aligned} 0 &= \delta H_x = H_{xx}\,\delta x + H_{xy}\,\delta y + d\lambda \\ 0 &= \delta H_y = H_{yx}\,\delta x + H_{yy}\,\delta y + d\mu. \end{aligned}$$

In the intervals of T_1 they are given by

$$(13) \qquad \begin{aligned} 0 &= \delta H_x = H_{xx}\,\delta x \qquad\qquad + d\lambda \\ 0 &= \qquad\qquad\qquad\qquad \delta y. \end{aligned}$$

since $y = 0$ on these intervals. In the intervals of T_2 they are given by

$$(14) \qquad \begin{aligned} 0 &= \qquad\quad \delta x \\ 0 &= \delta H_y = \qquad\quad H_{yy}\,\delta y + d\mu \end{aligned}$$

since $x = 0$. Finally in the intervals of T_3

$$(15) \qquad \begin{aligned} 0 &= \qquad \delta x \\ 0 &= \qquad\qquad \delta y. \end{aligned}$$

Hence we have equations of the form

$$(16) \qquad \begin{aligned} b_{11}(t)\,\delta x + b_{12}(t)\,\delta y + d\lambda &= 0 \\ b_{21}(t)\,\delta x + b_{22}(t)\,\delta y + d\mu &= 0 \end{aligned}$$

which can be solved for

$$(17) \qquad \begin{aligned} \delta x(t) &= a_{11}(t)\,d\lambda + a_{12}(t)\,d\mu \\ \delta y(t) &= a_{21}(t)\,d\lambda + a_{22}(t)\,d\mu \end{aligned}$$

where the a_{ij}'s are obtained from (12), (13), (14) or (15), depending on the interval.

We are now ready to calculate the total effects, dX and

dY, of the small changes $\delta x(t)$ and $\delta y(t)$. Integration over the entire interval [a,b], consisting of the four subsets T_0, T_1, T_2 and T_3, yields

$$\text{(18)} \qquad \begin{aligned} dX &= A_{11}\, d\lambda + A_{12}\, d\mu \\ dY &= A_{21}\, d\lambda + A_{22}\, d\mu \end{aligned}$$

where

$$dX = \int_a^b \delta x(t)\, dt, \qquad dY = \int_a^b \delta y(t)\, dt,$$

$$A_{ij} = \int_a^b a_{ij}(t)\, dt.$$

Now let us treat the differentials in (18) as differences to get

$$\text{(19)} \qquad \begin{aligned} A_{11}\, \Delta\lambda + A_{12}\, \Delta\mu &= \Delta X = 1 - X \\ A_{21}\, \Delta\lambda + A_{22}\, \Delta\mu &= \Delta Y = 1 - Y. \end{aligned}$$

These equations (19) can now be solved for $\Delta\lambda$ and $\Delta\mu$; thus we determine new values of λ and μ which tend to drive the solution toward admissibility.

Let us repeat this. We compute another trial solution using the new values of λ and μ. Repeat this sequence until the "errors" dX and dY become sufficiently small and the solution is admissible, within the required accuracy.

It is convenient to define an error term which can be used to check the convergence toward admissibility. Let the error, E, be defined as

$$\text{(20)} \qquad E = (1 - X)^2 + (1 - Y)^2 = \Delta X^2 + \Delta Y^2.$$

As the error goes to zero the solutions approach admissibility since both X and Y must be approaching unity and the differences, ΔX and ΔY, must approach zero.

An important feature of this differential method is that

each solution is an optimal (maximizing and minimizing) solution for the particular values of λ and μ, or endpoints X and Y, considered. The successive iterations drive x_0 and y_0 to admissibility, thus yielding the min-max (or saddle value) solution which satisfies the given constraints (2) and (3).

4. Second degree polynomial examples.

In order to test the method a computer program was written, using FORTRAN language, and run on a CDC 1604 computer. The form of this program and a sample solution, which will be explained in detail, appear in the Appendix. The program was set up first for an integrand of the type

$$K(x,y,t) = (y + At + B)^2 + xyt - (x + Ct + D)^2. \tag{21}$$

The parameters A, B, C and D were assigned various sets of integer values to test the general applicability of the method. In every case the interval [a,b] was chosen to be [0,1].

The choice of initial values of the Lagrange multipliers, λ and μ, is at times a minor problem in getting started. If the initial values yield only negative values of x(t) and y(t) over the interval [a,b], as determined by equations

$$H_x = K_x + \lambda = 0 \quad \text{and} \tag{7}$$

$$H_y = K_y + \mu = 0, \tag{9}$$

then the procedure has no way of determining appropriate changes, $\Delta\lambda$ and $\Delta\mu$, of the Lagrange multipliers for a new iteration since each of the A_{ij}'s in equation

$$\begin{aligned} dX &= A_{11}\, d\lambda + A_{12}\, d\mu \\ dY &= A_{21}\, d\lambda + A_{22}\, d\mu \end{aligned} \tag{18}$$

will be zero. The solution of equations (7) and (9) at any point on the interval [a,b] with the requirement that both x and y be positive at that point should indicate appropriate initial values of λ and μ such that convergence and a solution will occur.

Now let us go through the method in detail for a specific

(22) $$K(x,y,t) = (y + 2t)^2 + xyt - (x + 3t)^2.$$

The Hamiltonian, H, is

(23) $$H = (y + 2t)^2 + xyt - (x + 3t)^2 + \lambda x + \mu y$$

and its second partial derivatives are

(24) $$\begin{aligned} H_{xy} &= t = K_{xy} \\ H_{xx} &= -2 = K_{xx} < 0 \\ H_{yy} &= 2 = K_{yy} > 0. \end{aligned}$$

The first partial derivatives of H are

(25) $$\begin{aligned} H_x &= -2x + ty - 6t + \lambda \\ H_y &= tx + 2y + 4t + \mu. \end{aligned}$$

We can see that K is strictly concave in x and strictly convex in y from (24). Upon setting $H_x = H_y = 0$ we can solve for

(26) $$\begin{aligned} x(t) &= \frac{-4t^2 - 12t - \mu t + 2\lambda}{t^2 + 4} \\ y(t) &= \frac{6t^2 - 8t - \lambda t - 2\mu}{t^2 + 4}. \end{aligned}$$

Recall that the maximizing function $x = x_0$ and the minimizing function $y = y_0$ satisfy

$$\begin{aligned} x_0 &= x(t) \quad \text{if } x(t) \geq 0 \\ &= 0 \quad \text{if } x(t) < 0 \\ y_0 &= y(t) \quad \text{if } y(t) \geq 0 \\ &= 0 \quad \text{if } y(t) < 0 \end{aligned}$$

when $x(t)$ and $y(t)$ satisfy equations (26).

To determine appropriate initial values of λ and μ in this problem, evaluate (26) at a particular value of t, say $t = 0$. Then $x(0) = 0.5\lambda$ and $y(0) = -0.5\mu$, and any values of

$\lambda > 0$ and $\mu < 0$ are satisfactory for an initial guess. If we take $\lambda = 10$ and $\mu = -10$, then $x(0) = 5$ and $y(0) = 5$.

Let us arbitrarily divide the interval $[0,1]$ into 100 equal parts of length 0.01 and calculate $x(t)$ and $y(t)$ by (26) for $t = 0, 0.01, 0.02, \cdots, 0.99, 1.00$. Meanwhile we should determine which of the four subsets of intervals, T_0, T_1, T_2 or T_3, we are in for each value of t in order to calculate the a_{ij}'s which are necessary for the determination of $\Delta\lambda$ and $\Delta\mu$ if the solution is not admissible. The a_{ij}'s are given: in T_0 $(x > 0, y > 0)$ by

$$(27)\qquad \begin{aligned} a_{11}(t) &= \frac{-K_{yy}}{K_{xx}K_{yy} - K_{xy}^2}, & a_{12}(t) &= \frac{K_{xy}}{K_{xx}K_{yy} - K_{xy}^2}, \\ a_{21}(t) &= \frac{K_{xy}}{K_{xx}K_{yy} - K_{xy}^2}, & a_{22}(t) &= \frac{-K_{xx}}{K_{xx}K_{yy} - K_{xy}^2}; \end{aligned}$$

in T_1 $(x > 0, y = 0)$ by

$$(28)\qquad \begin{aligned} a_{11}(t) &= \frac{-1}{K_{xx}}, \\ a_{12}(t) &= a_{21}(t) = a_{22}(t) = 0; \end{aligned}$$

in T_2 $(x = 0, y > 0)$ by

$$(29)\qquad \begin{aligned} a_{11}(t) &= a_{12}(t) = a_{21}(t) = 0, \\ a_{22}(t) &= \frac{-1}{K_{yy}}; \end{aligned}$$

and in T_3 $(x = 0, y = 0)$ by

$$(30)\qquad a_{11}(t) = a_{12}(t) = a_{21}(t) = a_{22}(t) = 0.$$

For this particular example we find that $T_0 = [0, 0.82]$, T_1 is empty, $T_2 = (0.82, 1.0]$ and T_3 is empty.

Let us integrate $x_0(t)$ and $y_0(t)$ using a trapezoidal or,

if more accuracy is desired, Runge-Kutta method to determine if they are admissible, i.e. they integrate to unity. They are not admissible since

$$\int_0^1 x_0 \, dt = 4.1225\cdots \quad \text{and} \quad \int_0^1 y_0 \, dt = 3.0644\cdots.$$

For each value of t on [0,1], a change in x, y, λ and μ according to

$$(17) \qquad \begin{aligned} \delta x &= a_{11}\, d\lambda + a_{12}\, d\mu \\ \delta y &= a_{21}\, d\lambda + a_{22}\, d\mu \end{aligned}$$

will retain the maximizing and minimizing properties of x_0 and y_0. The total changes over the interval [0,1] are

$$(18) \qquad \begin{aligned} dX &= A_{11}\, d\lambda + A_{12}\, d\mu \\ dY &= A_{21}\, d\lambda + A_{22}\, d\mu \end{aligned}$$

where

$$dX = \int_0^1 \delta x \, dt \, , \qquad dY = \int_0^1 \delta y \, dt \, ,$$

$$A_{ij} = \int_0^1 a_{ij} \, dt.$$

For first order approximations treat the differentials as differences and set

$$\Delta X = 1 - X = A_{11}\, \Delta\lambda + A_{12}\, \Delta\mu = -3.1225\cdots$$

$$\Delta Y = 1 - Y = A_{21}\, \Delta\lambda + A_{22}\, \Delta\mu = -2.0644\cdots.$$

Solve for

$$\Delta\lambda = -5.3532\cdots$$

$$\Delta\mu = 5.7408\cdots.$$

Repeat the calculations of x and y by (26) with

$$\lambda = 4.6468\cdots$$

$$\mu = -4.2592\cdots.$$

Continue the routine and repeat until ΔX and ΔY are acceptably

small.

Other polynomials solved by this procedure were:

$$K(x,y,t) = [1 + (t - .5)^2]y^2 + xyt - x^2, \tag{31}$$

$$K(x,y,t) = y^2\sin(t) - xy\cos(t) - x^2. \tag{32}$$

The last one was solved for the interval $[0,1]$ and also the interval $[0,\pi]$. Care must be taken to ensure that the integrand, K, maintains its strictly concave, strictly convex properties over the entire interval. The interval $[0,2\pi]$ cannot be used for (32) because $K_{yy} = 2\sin(t)$, which becomes negative on the interval $[\pi,2\pi]$.

5. Exponential search example.

The particular functional which kindled interest in the two-sided allocation problem was the integral

$$\int_a^b x[1 - e^{-gy}]\, dt$$

which represents the probability of detecting an object distributed as $x = x(t)$ over the interval $[a,b]$ by use of a search effort distributed as $y = y(t)$ over the same interval if the law of detection has the exponential-saturation form as in random search[7, 9]. The function $g = g(t)$ is a measure of the ease with which a unit of search conducted near t detects a target there and is considered known. For "standard" conditions, g may be considered unity.

To maximize this integral with respect to y it is necessary to minimize over y

$$I = \int_a^b x\, e^{-gy}\, dt. \tag{33}$$

This problem was posed and a method of solution given in Search and Screening, O.E.G. Report No. 56[9] and in a paper written by Bernard O. Koopman[7] who edited the above report. However, the maximization over x of

$$\min_y \int_a^b x\, e^{-gy}\, dt$$

was not considered.

Notice that this problem does not meet the requirement of strict concavity in x since $K_{xx} \equiv 0$. However the differential method produced a min-max solution and min-max strategies with no difficulty. Minor modifications to the FORTRAN program listed in the Appendix were made to fit it to this

problem. The function g was assumed to be of the form

$$g = A + Bt + Ct^2 + Dt^3$$

and required to be strictly greater than zero. The parameter A was given the values 0.01, 0.1, 1, 3 and the other parameters were either all zero or else one was set equal to ±0.9 until 17 different problems were solved. The smallest and largest values of g were 0.01 and 3.0. The number of iterations required to drive the error less than 10^{-16} was never more than six. The program compiled in 47 seconds and required about 10 seconds to complete a single solution.

It became apparent that this particular problem could be solved more easily than by the differential method used. Consider the Hamiltonian

$$H = x\, e^{-gy} + \lambda x + \mu y.$$

To minimize H over y(t) we solve

$$H_y = -gx\, e^{-gy} + \mu = 0.$$

We may solve for y, obtaining

$$(34) \qquad \begin{aligned} y(t) &= \frac{1}{g} \log_e \frac{gx}{\mu} \qquad && \text{if } g(t)x(t) > \mu \\ &= 0 && \text{if } g(t)x(t) < \mu. \end{aligned}$$

This is the solution obtained by Koopman[7, 9] for any fixed function x(t). We must also maximize H over x(t) and hence we have a second equation

$$H_x = e^{-gy} + \lambda = 0.$$

We must solve these simultaneously for x and y. We require then

$$y(t) = \frac{1}{g(t)} \log_e \frac{1}{-\lambda}$$

or $g(t)y(t)$ = constant. From (34) we see that gx must also be constant if gy is to be constant. Due to the requirements for admissibility it follows that

$$x(t) = y(t) = \frac{\gamma}{g(t)}$$

where γ is a constant to be determined. If the constant γ is assigned any reasonable positive number, one iteration will be sufficient to determine the min-max strategies (x^*, y^*) since the initial solution (x_0, y_0) can be made admissible by normalizing it. For example if

$$\int_a^b x_0 \, dt = X = \int_a^b y_0 \, dt$$

then

$$\int_a^b \frac{x_0(t)}{X} \, dt = 1 = \int_a^b \frac{y_0(t)}{X} \, dt$$

and

$$x^*(t) = \frac{x_0(t)}{X} = y^*(t).$$

All that remains is to show that the solution attained is truly a min-max solution in that x^* and y^* are min-max strategies. Consider the payoff

$$I(x^*, y^*) = \int_a^b x^* e^{-gy^*} dt = \int_a^b -\lambda x^* dt = -\lambda.$$

Observe that if gy^* remains constant over the interval $[a,b]$ then for all admissible x we have

$$I(x, y^*) = \int_a^b -\lambda x(t) \, dt = -\lambda = I(x^*, y^*).$$

Since the integrand, K, is strictly convex in y ($K_{yy} > 0$), it is obvious that for all admissible $y(t)$

$$I(x^*, y^*) \leq I(x^*, y).$$

If $x = x_1(t) \neq x^*(t)$, then there exists a $y_1^*(t)$ such

that

$$I(x_1, y_1^*) < I(x^*, y^*)$$

and $x \neq x^*(t)$ obviously does not maximize I. In conclusion

$$I(x_1, y_1^*) < I(x_1, y^*) = I(x^*, y^*) \leq I(x^*, y)$$

and (x^*, y^*) is a min-max strategy.

As a rough check to ensure that x^* was a maximizing strategy, equation (34) was used to calculate the minimizing strategies y^* for predetermined non-optimal strategies

$$x(t) = 1,$$

$$x(t) = 0.05 + 1.9t,$$

$$x(t) = 1.95 - 1.9t,$$

letting g(t) vary as in the min-max calculations previously. In every case the min-max solution payoff was greater than the minimizing solution payoff when x was fixed not equal to $x^*(t)$.

6. Conclusions.

The differential methods discussed in this thesis can be used to determine

$$\max_x \min_y \int_a^b K(x,y,t)\, dt$$

subject to

$$(35) \qquad \begin{aligned} x \geq 0, &\qquad \int_a^b x\, dt = X \\ y \geq 0, &\qquad \int_a^b y\, dt = Y \end{aligned}$$

where X and Y are given numbers and the integrand has certain properties. It must not be monotone increasing in x or monotone decreasing in y, for x and y large, over the interval [a,b]. The integrand, K, is assumed to be finite and real valued although it need not be continuous as long as the regions of discontinuity are known. Determination of maxima and minima is accomplished by comparison of the value of K at stationary extrema with its value at the discontinuities.

This thesis specifically covers the cases when K is twice differentiable, strictly concave in x ($K_{xx} < 0$), and strictly convex in y ($K_{yy} > 0$) for non-negative piecewise continuous x and y over the domain [a,b] of the parameter t. For this type of function, K, a min-max solution, $I(x^*(t), y^*(t))$, and a min-max strategy $(x^*(t), y^*(t))$ can be determined which satisfy the constraints (35).

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APPENDIX

First the symbols and terms used in the FORTRAN program will be correlated with those introduced in the text of the thesis; then a brief explanation of the program, the program itself, and a sample output will follow.

FORTRAN symbols	Text	Explanation
T	t	the independent variable defined on the interval $[0,1]$
XM	x_0	that x which maximizes H where $H = k + \lambda x + \mu y$
YM	y_0	that y which minimizes H
FUNF	K	$I = \int_0^1 K \, dt$ where $K = K(x,y,t)$
FXYF	K_{xy}	$\partial^2 K/\partial x \partial y$
FXXF	K_{xx}	$\partial^2 K/\partial x^2$
FYYF	K_{yy}	$\partial^2 K/\partial y^2$
A(1,1)	$a_{11}(t)$	from equation (17)
ATOT(1,2)	$\int_0^1 a_{12} \, dt$	from equation (18)
U	λ	Lagrange multiplier
V	μ	Lagrange multiplier
XTOT	X	$\int_0^1 x_0 \, dt$
YTOT	Y	$\int_0^1 y_0 \, dt$
T(I)	t	a particular value of t on $[0,1]$
X	$x(t)$	a solution of equation (26)
Y	$y(t)$	a solution of equation (26)
DX	δx	a small change of $x_0(t)$
DY	δy	a small change of $y_0(t)$
DU	$d\lambda$	a small change of λ

FORTRAN symbols	Text	Explanation
DV	$d\mu$	a small change of μ
DIFFX	dX	$1 - \int_0^1 x_0 \, dt$
DIFFY	dY	$1 - \int_0^1 y_0 \, dt$
PAYOFF	I(x,y)	$\int_0^1 K \, dt$

Items set off by a brace of CCCCCC's will generally change when the integrand, K, is changed.

After the maximum number of iterations, division of the interval [a,b], acceptable error and initial Lagrange multiplier values are determined, we commence the iterations. The interval [a,b] has been chosen to be [0,1]. The variable T(I) is defined in such a way that it takes on values from zero to one as the index, I, goes from 1 to 101. We now solve the first derivative equations (26) for X and Y, and determine which set of subintervals, T_3, T_2, T_1 or T_0, we are in so that XM, YM, A(1,1), A(1,2), A(2,1) and A(2,2) may be determined.

Since we are using a trapezoidal integration routine, we sum XM, YM, A(1,1), A(1,2), A(2,1) and A(2,2) as we calculate them and then multiply each total by the width of one interval, Δt, to determine the integrals. After the error is determined, pertinent values are printed and then if the solution is not admissible, within the acceptable error, changes of the Lagrange multipliers are computed using the subroutine at the end of the program. The iterations are repeated until the error is acceptable and the solution XM, YM is admissible. The payoff is now calculated using again the trapezoidal

integration method. The min-max solution (PAYOFF) and min-max strategies (XM, YM) are now printed.

Statements 150, 140 and 135 are used only if poor initial values of the Lagrange multipliers are chosen such that the matrix ATOT is singular (i.e. its determinant is zero). The printout is for information only.

The numbers at the end of the program are sample values of AA, B, C and D to be read into the program as data.

```
      PROGRAM MINMAX
      DIMENSION T(900), XM(900), YM(900), ERROR(50),
     1     A(2,2), ATOT(2,2)
C
C     INSERT THE FUNCTION TO BE INTEGRATED AND ITS
C     SECOND PARTIAL DERIVATIVES
CC
CCCC
CCCCCC
      FUNF(X,Y,T) = (Y+AA*T+B)**2+X*Y*T-(X+C*T+D)**2
      FXYF(X,Y,T) = T
      FXXF(X,Y,T) = -2.
      FYYF(X,Y,T) = 2.
CCCCCC
CCCC
CC
C     MAXIMUM NUMBER OF ITERATIONS = NR.  MUST AGREE WITH
C     ARRAY SIZE OF ERROR
      NR = 50
C
C     NUMBER OF INTERVALS OVER WHICH TO EVALUATE
C     AND INTEGRATE
      NIPLUS1 = 101
C
C     EPSILON IS THE ACCEPTABLE ERROR
      EPSILON = .1E-15
C
      WRITE OUTPUT TAPE 6, 200
  200 FORMAT(1H1,////////31X, 18HSOLUTION OF MINMAX////)
      CALL TIME(6)
C
CC
CCCC
CCCCCC
    2 READ INPUT TAPE 5, 8,AA, B, C, D
    8 FORMAT(4F10.5)
      WRITE OUTPUT TAPE 6, 210, AA, B, C, D
  210 FORMAT(16HK(X,Y,T) = (Y + ,F8.5,5H*T + ,F8.5,
     1       19H)**2 + X*Y*T - (X +,F8.5,5H*T + ,F8.5,
     2       4H)**2///)
CCCCCC
CCCC
CC
      WRITE OUTPUT TAPE 6, 220
  220 FORMAT(3HRUN,10X,1HU,16X,1HV,15X,4HXTOT,13X,4HYTOT,
     1        14X,5HERROR//)
C
C
C     SET THE INITIAL VALUES OF THE LAGRANGE MULTIPLIERS
C
CC
CCCC
CCCCCC
      U =  10.
      V = -10.
CCCCCC
CCCC
CC
C
      FNI = NIPLUS1 - 1
C
C     COMMENCE ITERATIONS
C
      DO 100  L = 1, NR
      XTOT = 0.
      YTOT = 0.
      DO 5  M = 1,2
      DO 5 N = 1,2
    5 ATOT(M,N) = 0.
C
C     COMMENCE CALCULATIONS OVER THE INTERVAL (0,1)
C
```

```
      DO 99 I = 1, NIPLUS1
      FI = I - 1
      T(I) = FI/FNI
C
C     SOLVE FIRST DERIVATIVE EQUATIONS FOR X,Y
CC
CCCC
CCCCCC
      DENOM = T(I)**2 + 4.
      Y = (T(I)*(2.*C*T(I) - 4.*AA + 2.*D - U) - 4.*B
     1   - 2.*V)/DENOM
      X = (-T(I)*(2.*AA*T(I) + 2.*B + 4.*C + V) - 4.*D
     1     + 2.*U)/DENOM
CCCCCC
CCCC
CC
C
      IF (X) 10,20,20
   10 IF (Y) 15, 25, 25
   20 IF (Y) 35, 45, 45
C
C     BOTH X AND Y NEGATIVE
C
   15 XM(I) = 0.
      YM(I) = 0.
      A(1,1) = 0.
      A(1,2) = 0.
      A(2,1) = 0.
      A(2,2) = 0.
      GO TO 50
C
C     X NEGATIVE AND Y NON-NEGATIVE
C
   25 XM(I) = 0.
      YM(I) = Y
      A(1,1) = 0.
      A(1,2) = 0.
      A(2,1) = 0.
      A(2,2) = -1./FYYF(XM(I), YM(I), T(I))
      GO TO 50
C
C     X NON-NEGATIVE AND Y NEGATIVE
C
   35 XM(I) = X
      YM(I) = 0.
      A(1,1) = -1./FXXF(XM(I), YM(I), T(I))
      A(1,2)= 0.
      A(2,1) = 0.
      A(2,2) = 0.
      GO TO 50
C
C     BOTH X AND Y POSITIVE
C
   45 XM(I) = X
      YM(I) = Y
      DET = FXXF(XM(I),YM(I),T(I))*FYYF(XM(I),YM(I),T(I))
     1      -FXYF(XM(I), YM(I), T(I))**2
      A(1,1) = -FYYF(XM(I), YM(I), T(I))/DET
      A(1,2) =  FXYF(XM(I), YM(I), T(I))/DET
      A(2,1) = A(1,2)
      A(2,2) = -FXXF(XM(I), YM(I), T(I))/DET
C
C
C     DX = A(1,1)*DU + A(1,2)*DV
C     DY = A(2,1)*DU + A(2,2)*DV
C
C
C     TRAPEZOIDAL INTEGRATION ROUTINE
C
   50 IF(I-1)52,52,54
   52 XTOT=XTOT+XM(I)/2.
      YTOT=YTOT+YM(I)/2.
```

```
      DO 53 M=1,2
      DO 53 N=1,2
   53 ATOT(M,N)=ATOT(M,N)+A(M,N)/2.
      GO TO 99
   54 IF(I-NIPLUS1)55,52,52
   55 XTOT=XTOT+XM(I)
      YTOT=YTOT+YM(I)
      DO 56 M=1,2
      DO 56 N = 1,2
   56 ATOT(M,N)=ATOT(M,N)+A(M,N)
      GO TO 99
   99 CONTINUE
      XTOT = XTOT/FNI
      YTOT = YTOT/FNI
      DO 60 M = 1,2
      DO 60  N= 1,2
   60 ATOT(M,N) = ATOT(M,N)/FNI
C
C
C     DETERMINE ERROR
C
      DIFFX = 1. - XTOT
      DIFFY = 1. - YTOT
      ERROR(L) = DIFFX**2 + DIFFY**2
      WRITE OUTPUT TAPE 6, 230,L,U,V,XTOT,YTOT,ERROR(L)
  230 FORMAT(I2,3X,F14.8,3X,F14.8,5X,F12.8 ,5X,F12.8,5X,
     1       E14.8)
      IF(ERROR(L) - EPSILON) 105, 95, 95
C
C     CALL SUBROUTINE TO SOLVE MATRIX FOR DU AND DV
C
   95 CALL SOLVE(ATOT(1,1),ATOT(1,2),ATOT(2,1),ATOT(2,2),
     1             DIFFX, DIFFY, DU, DV, IND)
      GO TO (97,150),IND
   97 U = U + DU
  100 V = V + DV
C
C     END OF A RUN
C     REPEAT CALCULATIONS USING THE NEW VALUES OF U AND V
C
C
C     BEST U, V, XM(I), YM(I) ARE NOW STORED FOR EACH T(I)
C     INTEGRATE FUNF(XM(I), YM(I), T(I))
C
  105 PAYOFF=0.
      DO 70 I=1,NIPLUS1
      IF(I-1)62,62,64
   62 PAYOFF = PAYOFF + FUNF(XM(I), YM(I), T(I))/2.
      GO TO 70
   64 IF(I-NIPLUS1)65,62,62
   65 PAYOFF = PAYOFF + FUNF(XM(I), YM(I), T(I))
   70 CONTINUE
      PAYOFF=PAYOFF/FNI
C
C     END OF COMPUTATION
C
      WRITE OUTPUT TAPE 6,110,PAYOFF
  110 FORMAT(//9HPAYOFF = ,E14.8//)
      WRITE OUTPUT TAPE 6,130 (T(I),XM(I),YM(I),
     1                          I = 1,NIPLUS1)
  130 FORMAT(21X,1HT,18X,2HXM,20X,2HYM,//(17X,F8.5,10X,
     1       F12.9,10X,F12.9))
      GO TO 160
  150 WRITE OUTPUT TAPE 6,140
  140 FORMAT(5X,31HDETERMINANT OF THE MATRIX ATOT ,
     1       11HEQUALS ZERO///)
      WRITE OUTPUT TAPE 6,135,XTOT,YTOT,U,V,ATOT(1,1),
     1                         ATOT(1,2),ATOT(2,1),ATOT(2,2)
  135 FORMAT(7HXTOT = ,E12.6,10X,7HYTOT = ,E12.6/3X,4HU = ,
     1       E12.6,13X,4HV = ,E12.6//1X,6HA11 = ,E12.6,
     2       11X,6HA12 = ,E12.6/1X,6HA21 = ,E12.6,11X,
     3       6HA22 = ,E12.6/)
```

```
  160 CALL TIME(6)
      WRITE OUTPUT TAPE 6, 240
  240 FORMAT(1H1)
      GO TO 2
C     READ DATA FOR ANOTHER PROBLEM
      END
C
C
      SUBROUTINE SOLVE(A11,A12,A21,A22,B1,B2,X1,X2,IND)
C
C     SOLVES THE MATRIX AX=B FOR X
C
      D=A11*A22-A12*A21
      IF(D) 1,12,1
    1 X1=A22/D*B1-A12/D*B2
      X2=-A21/D*B1+A11/D*B2
      IND=1
      RETURN
   12 IND=2
      X1=0.
      X2=0.
      RETURN
      END
      END

    2.          0.          3.          0.
    3.          1.          4.          1.
    3.         -1.          3.          2.
    5.          4.          3.          2.
```

SOLUTION OF MINMAX

1 MINUTES --- 36 SECONDS

K(X,Y,T) = (Y + 2.00000*T + .00000)**2 + X*Y*T - (X + 3.00000*T + .00000)**2

RUN	U	V	XTOT	YTOT	ERROR
1	10.00000000	-10.00000000	4.12249158	3.06443499	.14011845E+02
2	4.64680696	-4.25921316	1.03996801	1.00000000	.15974421E-02
3	4.54908647	-4.24278961	1.00020455	1.00328803	.10852971E-04
4	4.54991554	-4.23605753	1.00000000	1.00007422	.55091278E-08
5	4.54994568	-4.23590744	1.00000000	1.00000127	.16130576E-11
6	4.54994620	-4.23590487	1.00000000	1.00000002	.47012785E-15
7	4.54994621	-4.23590482	1.00000000	1.00000000	.17957098E-18

PAYOFF = -.19464557E+01

T	XM	YM
.00000	2.274973104	2.117952412
.01000	2.255406481	2.086675379
.02000	2.235529075	2.055597121
.03000	2.215343937	2.024722252
.04000	2.194354210	1.994055327
.05000	2.174063125	1.963600833
.06000	2.152973999	1.933363192
.07000	2.131590240	1.903346753
.08000	2.109915336	1.873555798
.09000	2.087952858	1.843994533
.10000	2.065706458	1.814667089
.11000	2.043179867	1.785577519
.12000	2.020376892	1.756729798
.13000	1.997301412	1.728127820
.14000	1.973957381	1.699775395
.15000	1.950348822	1.671676250
.16000	1.926479826	1.643834025
.17000	1.902354547	1.616252275
.18000	1.877977205	1.588934463
.19000	1.853352080	1.561883964
.20000	1.828483510	1.535104061

T	XM	YM
.21000	1.803375888	1.508597943
.22000	1.778033662	1.482368709
.23000	1.752461330	1.456419359
.24000	1.726663440	1.430752799
.25000	1.700644583	1.405371839
.26000	1.674409398	1.380279190
.27000	1.647962561	1.355477466
.28000	1.621308789	1.330969181
.29000	1.594452833	1.306756751
.30000	1.567399477	1.282842490
.31000	1.540153539	1.259228613
.32000	1.512719861	1.235917234
.33000	1.485103314	1.212910365
.34000	1.457308790	1.190209917
.35000	1.429341201	1.167817701
.36000	1.401205480	1.145735425
.37000	1.372906572	1.123964696
.38000	1.344449437	1.102507018
.39000	1.315839044	1.081363798
.40000	1.287080371	1.060536337
.41000	1.258178401	1.040025839
.42000	1.229138119	1.019833407
.43000	1.199964513	.999960041
.44000	1.170662566	.980406647
.45000	1.141237260	.961174028
.46000	1.111693568	.942262891
.47000	1.082036457	.923673844
.48000	1.052270880	.905407400
.49000	1.022401778	.887463976
.50000	.992434077	.869843892
.51000	.962372685	.852547377
.52000	.932222490	.835574564
.53000	.901988360	.818925496
.54000	.871675137	.802600124
.55000	.841287639	.786598311
.56000	.810830656	.770919828
.57000	.780308947	.755564362
.58000	.749727242	.740531511
.59000	.719090237	.725820792
.60000	.688402594	.711431633
.61000	.657668936	.697363386
.62000	.626893852	.683615317
.63000	.596081888	.670186617
.64000	.565237550	.657076395
.65000	.534365302	.644283688
.66000	.503469564	.631807455
.67000	.472554709	.619646584
.68000	.441625066	.607799889

T	XM	YM
.69000	.410684914	.596266116
.70000	.379738483	.585043942
.71000	.348789956	.574131977
.72000	.317343459	.563528766
.73000	.286903072	.553232790
.74000	.255972817	.543242469
.75000	.225056665	.533556162
.76000	.194158529	.524172171
.77000	.163282268	.515088738
.78000	.132431685	.506304054
.79000	.101610524	.497816255
.80000	.070322473	.439623423
.81000	.040071159	.481723592
.82000	.009360151	.474114750
.83000	.000000000	.466794833
.84000	.000000000	.459761738
.85000	.000000000	.453013313
.86000	.000000000	.446547369
.87000	.000000000	.440361674
.88000	.000000000	.434453959
.89000	.000000000	.428821920
.90000	.000000000	.423463214
.91000	.000000000	.418375468
.92000	.000000000	.413556276
.93000	.000000000	.409003201
.94000	.000000000	.404713779
.95000	.000000000	.400685517
.96000	.000000000	.396915899
.97000	.000000000	.393402381
.98000	.000000000	.390142400
.99000	.000000000	.387133371
1.00000	.000000000	.384372688

1 MINUTES --- 44 SECONDS